Distortion Analysis for Energy Measurement Equipment

(Performance estimation based upon simple distortion prototypes)

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Abstract: The intended audiences are ISO 17025 Calibration Laboratories, ISO 9001 Meter Manufacturers, Electric utilities, and Organizations involved with electricity measurement. Participants will gain insights in recognizing the need to better define methods of energy measurement in order to have a better understanding on why errors may be present during power or energy calibration of equipment and how various error sources effect measurements.

This paper will show the possible differences and potential influences associated with industry accepted measurement algorithms associated to electricity measurement. Radian Research is a world leading manufacturer of primary energy reference standards for electric power and has done extensive studies to help ensure that accurate testing of electrical power and energy measurement is maintained over a broad range of testing conditions.

Key words: Energy Measurement, Measurement Algorithms, Error Influence, Harmonic Distortion.

1. Introduction

Establishing how a given electricity measurement device measures active, reactive, and apparent energy is of paramount concern to electric utility companies, manufacturers of energy meters, energy-reference-standards, and energy meter test systems. Although an energy meter can be identified as belonging to a particular accuracy class, it is not generally given under what real-world ambient load conditions the measurements of a meter are applicable. With the introduction of digital technology and digital measurement techniques, there arises the introduction of a purely mathematical abstraction of the energy measurement process within modern meters. Therefore, it is no longer possible to predict how a meter will respond by only examining the analog component; the digital component is a key element within the measurement procedure as well (which is well hidden from everyone but the designer). It can be shown that different meters tend to contain different solutions to the energy measurement equations. Various meters can be grouped together, for they produce measurements that are identical to one another under identical environments conditions. While others form separate groups, each group establishing an equivalent accuracy class of measurements among themselves under the same environmental conditions. Establishing a procedure or method to determine where a given meter fits into a “reference set of solutions to the energy measurement equations” can be instrumental to meter manufacturers in establishing under what environmental conditions the accuracy of their meter holds good. It should be clearly noted that the goal of this paper is not to be critical of the measurement algorithm used – much progress has been made over the years in the advancement of electronic energy meter performance. Instead, the goal is to demonstrate the need to define the measurement approach in order to properly compare its performance to a reference. Unfortunately the measurement approaches are not consistent between devices and only when the methods are defined can the device accuracy be compared to a reference.

The purpose of this study is to investigate the error behavior of the set of classic energy metrics in the presence of particular distortions. The set of ***metrics*** under investigation have the forms:

- RMS Voltage :

$$V = \sqrt{\frac{1}{kT}\int_{\tau}^{\tau+kT} v^2 dt} \quad (1)$$

- RMS Voltage Squared:
$$V^2 = \frac{1}{kT}\int_{\tau}^{\tau+kT} v^2 dt \quad (2)$$
- RMS Current:
$$I = \sqrt{\frac{1}{kT}\int_{\tau}^{\tau+kT} i^2 dt} \quad (3)$$
- RMS Current Squared:
$$I^2 = \frac{1}{kT}\int_{\tau}^{\tau+kT} i^2 dt \quad (4)$$
- Arithmetic Apparent Power (VA):
$$S_A = VI \quad (5)$$
- Vector Apparent Power (VA):
$$S_{\rightarrow} = \sqrt{P^2 + Q^2} \quad (6)$$
- Active Power (W):
$$P = \frac{1}{kT}\int_{\tau}^{\tau+kT} vidt \quad (7)$$
- Integral Reactive Power (VAR):
$$Q_{\int} = \frac{1}{kT}\int_{\tau}^{\tau+kT} v_{\perp} idt \text{ where:}$$
$$v_{\perp} = \omega\int v^2 dt \quad (8)$$
- Shifted Reactive Power (VAR):
$$Q_{\pi/2} = \frac{1}{kT}\int_{\tau}^{\tau+kT} v_{\perp} idt$$
where: $v_{\perp} = v(t + \frac{\pi}{2\omega})$ (9)
- RMS Reactive Power (VAR):
$$Q_{rms} = \sqrt{S^2 - P^2} \quad (10)$$
- Power Factor:
$$PF = \frac{P}{S} \quad (11)$$

Where: T is the fundamental Period $T = \frac{1}{f}$, f is the fundamental frequency

τ is integration start time.

$k = 1,2,3,...$ the number of cycles integration is taken over.

$v(t)$ the in-phase voltage

$v_{\perp}(t)$ the quadrature-phase voltage

$i(t)$ the current.

$\omega = 2\pi f$

This study introduces a particular type of distortion and ask the question: due to the presence of a particular ***distortion D***, and given a set of ***metrics μ***, what are their expectations (their predicted value in the presence of a given distortion ***D***), i.e. E(μ, | ***D***).
In addition, what are the errors of the metrics relative to pure sinusoidal conditions (distortion free).

This study investigates three distortion prototypes.

- ***D_0***: Zero-mean additive Gaussian noise on the voltage and current axes.
- ***D_1***: An m^th order harmonic on the Voltage axis and an n^th order harmonic on Current axis were the orders can be the same or differ.

2. Power Measurements Derived from a Measurement Basis

Before proceeding, a brief analysis of power estimates will be investigated that assumes no a priori knowledge concerning the nature of the distortion on the voltage and current axes. This is accomplished by measuring a set of fundamental metrics and then deriving others from this measurement basis. After the additional estimates are derived, their errors can be directly computed.

Let the following define the measurement basis:

Measured RMS Voltage Squared: V^2

Measured RMS Current Squared: I^2

Measured Phase Angle: θ

And their ground truth counterparts:

True RMS Voltage: V_0^2

True RMS Current: I_0^2

True Phase Angle: θ_0

We take as the error equations for the measurement basis the following:

$$\xi_{V^2} = \frac{V^2 - V_0^2}{V_0^2} \quad \text{And} \quad V^2 = V_0^2\left(1 + \xi_{V^2}\right) \quad (12)$$

$$\xi_{I^2} = \frac{I^2 - I_0^2}{I_0^2} \quad \text{And} \quad I^2 = I_0^2\left(1 + \xi_{I^2}\right) \quad (13)$$

$$\xi_{\theta} = \frac{\theta - \theta_o}{2\pi} \quad \text{And} \quad \theta = \theta_o + 2\pi\xi_{\theta} \quad (14)$$

ξ_{V^2} , ξ_{I^2} , ξ_θ, are the normalized errors for our measurement basis.

Next, the expected estimates for: V , I , S , P , Q , PF will be derived based upon this measurement basis. This initial analysis assumes sinusoidal forms for these metrics, which is all that can be managed with the measurement basis given. This will be treated as starting point for the rest of the analysis. As the analysis progress more general non-sinusoidal forms will be introduced. Useful information about the error behavior of the listed metrics will be illustrated for this initial case.

2.1. RMS Voltage and RMS Current

RMS voltage and current is simply derived form the square roots of the V^2 and I^2:

$V = V_0\sqrt{1+\xi_{v^2}}$ and $\xi_V = \dfrac{V-V_0}{V_0} = \sqrt{1+\xi_{v^2}} - 1$

$I = I_0\sqrt{1+\xi_{I^2}}$ and $\xi_I = \dfrac{I-I_0}{I_0} = \sqrt{1+\xi_{I^2}} - 1$

2.2. VA based upon $S = VI$

VA is derived from the RMS voltage and current computed in section 2.1 above:

$S = VI = V_o I_o \sqrt{(1+\xi_{v^2})(1+\xi_{I^2})}$ and the VA error is then determined to be:

$$\xi_S = \frac{S-S_0}{S_0} = \frac{VI - V_o I_o}{V_o I_o}$$

$$= \frac{V_o I_o \sqrt{(1+\xi_{v^2})(1+\xi_{I^2})} - V_o I_o}{V_o I_o}$$

$$= \sqrt{(1+\xi_{v^2})(1+\xi_{I^2})} - 1 \qquad (15)$$

2.3. Watt based upon $P = VI\cos(\theta)$

The Watt measurement is expressed as:

$$P = VI\cos(\theta)$$
$$= V_o I_o \sqrt{(1+\xi_{v^2})(1+\xi_{I^2})}\cos(\theta_o + 2\pi\xi_\theta) \qquad (16)$$

The error equation for Watt is expressed as follows:

$$\xi_P = \frac{P-P_0}{P_0} = \frac{VI\cos(\theta) - V_o A_o\cos(\theta_o)}{V_o A_o \cos(\theta_o)}$$

$$= \frac{V_o I_o \sqrt{(1+\xi_{v^2})(1+\xi_{I^2})}\cos(\theta_o + 2\pi\xi_\theta) - V_o I_o\cos(\theta_o)}{V_o I_o\cos(\theta_o)}$$

$$= \frac{\sqrt{(1+\xi_{v^2})(1+\xi_{I^2})}\cos(\theta_o + 2\pi\xi_\theta) - \cos(\theta_o)}{\cos(\theta_o)}$$

$$= \frac{(1+\xi_v)(1+\xi_I)(\cos(\theta_o)\cos(2\pi\xi_\theta) - \sin(\theta_o)\sin(2\pi\xi_\theta)) - \cos(\theta_o)}{\cos(\theta_o)} =$$

$$\xi_P = \sqrt{(1+\xi_{v^2})(1+\xi_{I^2})}\begin{pmatrix}\cos(2\pi\xi_\theta) \\ -\tan(\theta_o)\sin(2\pi\xi_\theta)\end{pmatrix} - 1 \qquad (17)$$

2.4. VAR based upon $Q = VI\sin(\theta)$

The VAR measurement is expressed as:

$$Q = VI\sin(\theta) = V_o I_o \sqrt{(1+\xi_{v^2})(1+\xi_{I^2})}\sin(\theta_o + 2\pi\xi_\theta)$$

The error equation for VAR is expressed as follows:

$$\xi_{VAR} = \frac{V_{rms}A_{rms}\sin(\theta) - V_o A_o \sin(\theta_o)}{V_o A_o \sin(\theta_o)}$$

$$= \frac{V_o A_o \sqrt{(1+\xi_{v^2})(1+\xi_{A^2})}\sin(\theta_o + 2\pi\xi_\theta) - V_o A_o\sin(\theta_o)}{V_o A_o\sin(\theta_o)}$$

$$= \frac{(1+\xi_v)(1+\xi_A)\sin(\theta_o + 2\pi\xi_\theta) - \sin(\theta_o)}{\sin(\theta_o)} \backslash$$

$$= \frac{(1+\xi_v)(1+\xi_A)\begin{pmatrix}\sin(\theta_o)\cos(2\pi\xi_\theta) \\ +\cos(\theta_o)\sin(2\pi\xi_\theta)\end{pmatrix} - \sin(\theta_o)}{\sin(\theta_o)} =$$

$$\xi_{VAR} = \sqrt{(1+\xi_{v^2})(1+\xi_{A^2})}\begin{pmatrix}\cos(2\pi\xi_\theta) \\ +\cot(\theta_o)\sin(2\pi\xi_\theta)\end{pmatrix} - 1 \qquad (18)$$

2.5. Power Factor

The power factor follows from its definition:

$$PF = \frac{P}{S} = \frac{V_o I_o \sqrt{(1+\xi_{V^2})(1+\xi_{I^2})}\cos(\theta_o + 2\pi\xi_\theta)}{V_o I_o \sqrt{(1+\xi_{V^2})(1+\xi_{I^2})}} \quad (19)$$

$$= \cos(\theta_o + 2\pi\xi_\theta)$$

The corresponding error for power factor is:

$$\xi_{PF} = \frac{\cos(\theta_o + 2\pi\xi_\theta) - \cos(\theta_o)}{\cos(\theta_o)} \quad (20)$$

$$= \cos(2\pi\xi_\theta) - \tan(\theta_o)\sin(2\pi\xi_\theta) - 1$$

2.6. Conclusion regarding measurement basis

It is important to keep in mind that all the metrics derived from the measurement basis inherit the errors from the measurements basis.

3. *D₀*: Zero-mean additive Gaussian noise on the voltage and current axes

Assume the voltage and current are corrupted by a zero mean Gaussian noise distribution. The voltage and current signals as seen by the measurement device are defined as follows:

$$\tilde{v}(t) = v(t) + \eta_v(t) \quad \text{and} \quad \tilde{i}(t) = i(t) + \eta_i(t)$$

Where: $\tilde{v}(t)$ and $\tilde{i}(t)$ are the potential and current as seen by measurement device.

$v(t)$ and $i(t)$ are the true potential and current.

$\eta_v(t)$ and $\eta_i(t)$ are the zero mean additive white noise sources.

3.1. RMS voltage and RMS current in the presence distortion *D₀*

Let V_0 and I_0 represent the ***true*** RMS voltage and RMS current integrated over cycle T:

$$V_0 = \sqrt{\frac{1}{kT}\int_\tau^{\tau+kT} v^2 dt} \quad \text{and} \quad I_0 = \sqrt{\frac{1}{kT}\int_\tau^{\tau+kT} i^2 dt}$$

The variance of the noise distributions can be defined relative to the ***true*** RMS voltage and RMS current squared by:

$$\frac{1}{kT}\int_\tau^{\tau+kT} \eta_v^2 dt = V_o^2\sigma_V^2 \quad \text{and} \quad \frac{1}{kT}\int_\tau^{\tau+kT} \eta_i^2 dt = I_o^2\sigma_I^2$$

The RMS voltage squared as seen by the measurement device is given by:

$$\frac{\int_\tau^{\tau+kT} \tilde{v}^2 dt}{kT} = \frac{\int_\tau^{\tau+kT} (v+\eta_v)^2 dt}{kT}$$

$$= \frac{\int_\tau^{\tau+kT} v^2 dt + 2\int_\tau^{\tau+kT} v\eta_v dt + \int_\tau^{\tau+kT} \eta_v^2 dt}{kT} \quad (21)$$

$$= \frac{\int_\tau^{\tau+kT} v^2 dt + \int_\tau^{\tau+kT} \eta_v^2 dt}{kT}$$

Since $\eta_v(t)$ is zero mean and independent of v, then: $\int_\tau^{\tau+kT} v\eta_v dt = 0$. The expectation of this product integrated over a period T: $E(v\eta_v) = E(v)E(\eta_v) = 0$.

The measure RMS Voltage Squared will be:

$V^2 = V_o^2 + V_o^2\sigma_V^2 = V_o^2(1+\sigma_V^2)$ and

$$\xi_{V^2} = \frac{V^2 - V_0^2}{V_o^2} = \frac{V_o^2(1+\sigma_V^2) - V_o^2}{V_o^2} = \sigma_V^2 \quad (22)$$

Integrating similarly, the RMS Current Squared measurement will be:

$I^2 = I_o^2 + I_o^2\sigma_I^2 = I_o^2(1+\sigma_I^2)$ and

$$\xi_{I^2} = \frac{I^2 - I_0^2}{I_o^2} = \frac{I_o^2(1+\sigma_I^2) - I_o^2}{I_o^2} = \sigma_I^2 \quad (23)$$

The measure RMS Voltage and Current would be the square roots of V^2 and I^2:

$$V = V_0\sqrt{1+\sigma_V^2} \quad \text{and} \quad \xi_V = \frac{V - V_0}{V_0} = \sqrt{1+\sigma_V^2} - 1$$

$$I = I_0\sqrt{1+\sigma_I^2} \quad \text{and} \quad \xi_I = \frac{I - I_0}{I_0} = \sqrt{1+\sigma_I^2} - 1$$

3.2. Arithmetic VA in the presence distortion *D₀*

The arithmetic VA is determined from the RMS voltage and RMS current found in section 3.1:

$$S_A = V_0 I_0 \sqrt{\left(1+\sigma_V^2\right)\left(1+\sigma_I^2\right)} \quad \text{and}$$

$$\xi_{S_A} = \sqrt{\left(1+\sigma_V^2\right)\left(1+\sigma_I^2\right)} - 1 \qquad (24)$$

3.3. Vector VA in the presence of distortion D_0

Vector VA is a derived from the measurements of Watt and VAR. The defining equation for vector VA given in section 1.0 is:

$$S_{\rightarrow} = \sqrt{P^2 + Q^2} \qquad (25)$$

Substituting the measurements derived above:

$S_{\rightarrow} = \sqrt{P_0^2 + Q_0^2}$ and $\xi_{S_{\rightarrow}} = 0$

Note: Vector VA has no inherent error, whereas Arithmetic VA has an inherent error by the virtue of how it is derived. The Arithmetic VA can not avoid the contribution of the noise power density due to RMS voltage and current.

3.4. Watt in the presence distortion D_0

The integral definition of Watt with the voltage and current functions given in section 3.0 above is evaluated as follows:

$$P = \frac{1}{kT}\int_{\tau}^{\tau+kT} \tilde{v}\tilde{i}\,dt = \frac{1}{kT}\int_{\tau}^{\tau+kT} \left(v(t)+\eta_v(t)\right)\left(i(t)+\eta_i(t)\right)dt$$

$$= \frac{\int_{\tau}^{\tau+kT} vi dt + \int_{\tau}^{\tau+kT} v\eta_i dt + \int_{\tau}^{\tau+kT} i\eta_v dt + \int_{\tau}^{\tau+kT} \eta_v\eta_i dt}{kT} = \frac{1}{kT}\int_{\tau}^{\tau+kT} vidt$$

$$P = V_0 I_0 \cos(\theta_0) \qquad (26)$$

And the Watt error would be: $\xi_{Watt} = 0$.

For the ideal integrator, the Gaussian noise totally integrates out of the equation.

Note: the noise distributions are zero mean and independent from on another, therefore:

$$E(V_V\eta_A) = E(V_V)E(\eta_A) = 0\,,$$
$$E(A\eta_V) = E(A)E(\eta_V) = 0$$
$$E(\eta_V\eta_A) = E(\eta_V)E(\eta_A) = 0$$

3.5. Integral and Shifted VAR in the presence distortion D_0

Given: the quadrature-phase component of voltage $\tilde{v}_{\perp}$ is computed by integration or phase shifting. The integral or shifted definition of VAR with the voltage and current functions given in section 3.0 above is evaluated as follows:

$$Q_{\int} = \frac{1}{kT}\int_{\tau}^{\tau+kT} \tilde{v}_{\perp}\tilde{i}\,dt = \frac{1}{kT}\int_{\tau}^{\tau+kT} \left(v_{\perp}(t)+\eta_v(t)\right)\left(i(t)+i_A(t)\right)dt$$

$$= \frac{\int_{\tau}^{\tau+kT} v_{\perp} idt + \int_{\tau}^{\tau+kT} v_{\perp}\eta_i dt + \int_{\tau}^{\tau+kT} i\eta_v dt + \int_{\tau}^{\tau+kT} \eta_v\eta_i dt}{kT} = \frac{1}{kT}\int_{\tau}^{\tau+kT} v_{\perp} idt$$

$$Q_{\int} = V_0 I_0 \sin(\theta_0) \qquad (27)$$

And the VAR error would be $\xi_{Q_{\int}} = 0$

For the ideal integrator, the Gaussian noise totally integrates out of the equation.

Note: the noise distributions are zero mean and independent from on another, therefore:

$$E(V_{\perp}\eta_A) = E(V_{\perp})E(\eta_A) = 0\,,$$
$$E(A\eta_V) = E(A)E(\eta_V) = 0$$
$$E(\eta_V\eta_A) = E(\eta_V)E(\eta_A) = 0$$

3.6. RMS VAR in the presence of distortion D_0

Computing RMS VAR with respect to the power triangle yields:

$$Q_{rms} = \sqrt{S_A^2 - P^2}$$
$$= \sqrt{V_0^2 I_0^2\left(1+\sigma_V^2\right)\left(1+\sigma_I^2\right) - V_0^2 I_0^2\cos^2(\theta_0)}$$
$$Q_{rms} = V_0 I_0\sqrt{\left(1+\sigma_V^2\right)\left(1+\sigma_I^2\right) - \cos^2(\theta_0)} \qquad (28)$$
$$= V_0 I_0\sqrt{\sin^2(\theta_0) + \sigma_v^2 + \sigma_I^2 + \sigma_V^2\sigma_I^2}$$

$$\xi_{Q_{rms}} = \sqrt{1+\csc^2(\theta_0)\left(\sigma_V^2 + \sigma_I^2 + \sigma_V^2\sigma_I^2\right)} - 1 \qquad (29)$$

Note: Since Arithmetic VA has an error component related to the noise power density, then RMS VAR will inherit this error.

3.7. Power Factor in the presence of distortion D_0

3.7.1. Power Factor in the presence of distortion D_0: Arithmetic VA

If the power factor is derived using Arithmetic VA the expected estimate would be:

$$PF_A = \frac{P_0}{S_A} = \frac{V_0 I_0 \cos(\theta_0)}{V_0 I_0 \sqrt{(1+\sigma_V^2)(1+\sigma_I^2)}} = \frac{\cos(\theta_0)}{\sqrt{(1+\sigma_V^2)(1+\sigma_I^2)}} \quad (30)$$

The error of the estimate would be:

$$\xi_{PF} = \frac{1}{\sqrt{(1+\sigma_V^2)(1+\sigma_I^2)}} - 1 \quad (31)$$

3.7.2. Power Factor in the presence of distortion D_0: Vector VA

If the power factor is derived using Vector VA the expected estimate would be:

$PF_{\rightarrow} = \frac{P_0}{S_0} = \cos(\theta_0)$ and $\xi_{PF_{\rightarrow}} = 0$, no error in the estimate.

3.8. Conclusions regarding distortion *D_0: Z*ero-mean additive Gaussian noise

It is clear for the case of zero-mean additive Gaussian noise, a measurement device can not avoid including noise density into RMS Voltage and RMS Current measurements.

Secondly, if the integration does not involve the noise density, i.e. the noise is NOT squared in the measurement process, and the noise is zero mean, the noise integration will vanish. This is true in the case of Watt, Integral and Shifted VAR. Since the noise on the voltage and current axes are independent. Integration is an optimal estimator for these two metrics in the presence of zero-mean additive Gaussian noise.

Thus, metrics derived from RMS voltage and RMS current inherits noise power density components, thereby adding error to these derived metrics. We see that arithmetic VA, RMS VAR, and arithmetic PF, inherit errors from RMS Voltage and RMS Current. And since vector VA and PF are derived from Watt and Integral or Shifted VAR, they have no inherent error.

4. *D_1*: A m[th] order on voltage and a n[th] order on current – harmonics on all axes

Assume the voltage axis is corrupted by a harmonic component of order j and the current axis is corrupted by a harmonic component of order k. The voltage and current signals as seen by the measurement device are defined as follows:

$v(t) = \sqrt{2} V_0 \left(\sin(\omega t) + \beta \sin(m\omega t + \theta_m)\right)$ And

$i(t) = \sqrt{2} I_0 \left(\sin(\omega t + \theta_0) + \alpha \sin(n\omega_0 t + \theta_n)\right)$

Where: $v(t)$ and $i(t)$ are the current and potential as seen by the measurement device

V_0 and I_0 are the expected RMS current and potential.

m, n the harmonic number for the potential and current axes respectively

$\omega = 2\pi f$ where f is the fundamental frequency.

$THD_V = 100\beta^2$ or $\beta = \sqrt{THD_V / 100}$

$THD_I = 100\alpha^2$ or $\alpha = \sqrt{THD_I / 100}$

4.1. RMS voltage and current in the presence distortion *D_1*

Note: $\sin^2(\omega t + \theta_0) = \frac{1}{2}(1 - \cos(2\omega t + 2\theta_0))$

$\cos^2(\omega t + \theta_0) = \frac{1}{2}(1 + \cos(2\omega t + 2\theta_0))$

And

$$\int_\tau^{\tau+kT} \sin^2(\omega t + \theta_0) dt = \frac{1}{2}\int_\tau^{\tau+kT} (1 - \cos(2\omega t + 2\theta_0)) dt$$

$$= \frac{1}{2}\left(\int_\tau^{\tau+kT} dt - \int_\tau^{\tau+kT} \cos(2\omega t + 2\theta_0) dt\right)$$

$$= \Big|_\tau^{\tau+kT} \frac{1}{2}\left(t - \frac{1}{2\omega}\sin(2\omega t + 2\theta_0)\right)$$

$$= \frac{1}{2}\left(\begin{array}{l}\left(\tau + kT - \frac{1}{2\omega}\sin(2\omega t + 2\theta_0 + k2\pi)\right) \\ -\left(\tau - \frac{1}{2\omega}\sin(2\omega t + 2\theta_0)\right)\end{array}\right) = \frac{kT}{2} \quad (32)$$

And

$$\int_{\tau}^{\tau+kT}\cos^2(\omega t+\theta_0)dt=\frac{1}{2}\int_{\tau}^{\tau+kT}(1+\cos(2\omega t+2\theta_0))dt$$

$$=\frac{1}{2}(\int_{\tau}^{\tau+kT}dt+\int_{\tau}^{\tau+kT}\cos(2\omega t+2\theta_0)dt)$$

$$=\Big|_{\tau}^{\tau+kT}\frac{1}{2}(t+\frac{1}{2\omega}\sin(2\omega t+2\theta_0))$$

$$=\frac{1}{2}\left(\left((\tau+kT+\frac{1}{2\omega}\sin(2\omega t+2\theta_0+k2\pi)\right) -\left(\tau+\frac{1}{2\omega}\sin(2\omega t+2\theta_0)\right)\right)=\frac{kT}{2} \qquad (33)$$

The RMS current squared is given by:

$$\frac{1}{kT}\int_o^T i^2dt=\frac{\sqrt{2}I_0}{kT}\int_{\tau}^{\tau+kT}(\sin(\omega t+\theta_0)+\alpha\sin(n\omega t+\theta_n))^2dt$$

$$=\frac{2I_0^2}{kT}\left(\begin{array}{l}\int_{\tau}^{\tau+kT}\sin^2(\omega t+\theta_0)dt\\ +2\alpha\int_{\tau}^{\tau+kT}\sin(\omega t+\theta_0)\sin(n\omega t+\theta_n)dt\\ +\alpha^2\int_{\tau}^{\tau+kT}\sin^2(n\omega t+\theta_n)dt\end{array}\right) \qquad (34)$$

Evaluating each of the three integrals above, we have:

1): $\frac{2I_0^2}{kT}\int_{\tau}^{\tau+kT}\sin^2(\omega t+\theta_0)dt=I_0^2$

Because $\sin(\omega t)$, $\cos(\omega t)$,$\sin(n\omega t)$, and $\cos(n\omega t)$ are mutually orthogonal:

$$\int_{\tau}^{\tau+kT}\sin(\omega t)\sin(n\omega t)dt=0$$

$$\int_{\tau}^{\tau+kT}\sin(\omega t)\cos(n\omega t)dt=0$$

$$\int_{\tau}^{\tau+kT}\cos(\omega t)\sin(n\omega t)dt=0$$

$$\int_{\tau}^{\tau+kT}\cos(\omega t)\cos(n\omega t)dt=0$$

$$\int_{\tau}^{\tau+kT}\sin(\omega t+\theta_0)\sin(n\omega t+\theta_n)dt$$

$$=\int_{\tau}^{\tau+kT}\begin{array}{l}(\sin(\omega t)\cos(\theta_0)+\cos(\omega t)\sin(\theta_0))\\ (\sin(n\omega t)\cos(\theta_n)+\cos(n\omega t)\sin(\theta_n))dt\end{array}$$

Multiplying this out we have:

$$=\cos(\theta_0)\cos(\theta_n)\int_{\tau}^{\tau+kT}\sin(\omega t)\sin(n\omega t)dt+$$

$$\cos(\theta_0)\sin(\theta_k)\int_{\tau}^{\tau+kT}\sin(\omega t)\cos(n\omega t)dt+$$

$$\sin(\theta_0)\cos(\theta_n)\int_{\tau}^{\tau+kT}\cos(\omega t)\sin(n\omega t)dt+$$

$$\sin(\theta_0)\sin(\theta_n)\int_{\tau}^{\tau+kT}\cos(\omega t)\cos(n\omega t)dt=0$$

Therefore 2):

$$\frac{4I_0^2\alpha}{kT}\int_{\tau}^{\tau+kT}\sin(\omega t+\theta_0)\sin(n\omega t+\theta_n)dt=0 \qquad (35)$$

and finally 3): $\frac{2I_0^2\alpha^2}{kT}\int_{\tau}^{\tau+kT}\sin^2(n\omega t+\theta_n)dt=I_0^2\alpha^2$

$I^2=I_0^2(1+a^2)$ and

$$\xi_{I^2}=\frac{I^2-I_0^2}{I_o^2}=\frac{I_o^2\left(1+\alpha^2\right)-I_o^2}{I_o^2}=\alpha^2 \qquad (36)$$

Based on the analysis above for RMS voltage squared would be:

$V^2=\frac{1}{kT}\int_{\tau}^{\tau+kT}v^2dt=V_0^2(1+\beta^2)$ And

$$\xi_{V^2}=\frac{V^2-V_0^2}{V_o^2}=\frac{V_o^2\left(1+\beta^2\right)-V_o^2}{V_o^2}=\beta^2 \qquad (37)$$

The measure RMS Voltage and Current would be the square roots of V^2 and I^2:

$V=V_0\sqrt{1+\beta^2}$ and $\xi_V=\frac{V-V_0}{V_0}=\sqrt{1+\beta^2}-1$

$I=I_0\sqrt{1+\alpha^2}$ and $\xi_I=\frac{I-I_0}{I_0}=\sqrt{1+\alpha^2}-1$

4.2. Watt in the presence distortion D_1

The integral definition of Watt with the voltage and current functions given in section 1.0 above is evaluated as follows:

$$P=\frac{1}{kT}\int_{\tau}^{\tau+kT}vidt$$

$$=\frac{2V_0I_0}{kT}\int_{\tau}^{\tau+kT}\begin{array}{l}(\sin(\omega t)+\beta\sin(m\omega t+\theta_m))\\ (\sin(\omega t+\theta_0)+\alpha\sin(n\omega t+\theta_n))dt\end{array}$$

$$=\frac{2V_0I_0}{kT}\int_{\tau}^{\tau+kT}\sin(\omega t)\sin(\omega t+\theta_0)dt+$$

$$\frac{2V_0I_0\alpha}{kT}\int_{\tau}^{\tau+kT}\sin(\omega t)\sin(n\omega t+\theta_n)dt+ \qquad (38)$$

$$\frac{2V_0I_0\beta}{kT}\int_{\tau}^{\tau+kT}\sin(m\omega t+\theta_m)\sin(\omega t+\theta_0)dt+$$
$$\frac{2V_0I_0\alpha\beta}{kT}\int_{\tau}^{\tau+kT}\sin(m\omega t+\theta_m)\sin(n\omega t+\theta_m)dt$$

Evaluating each of these integrals:

$$\int_{\tau}^{\tau+kT}\sin(\omega t)\sin(\omega t+\theta_0)dt=\frac{kT\cos(\theta_0)}{2}$$
$$\int_{\tau}^{\tau+kT}\sin(\omega t)\sin(n\omega t+\theta_n)dt=0$$

Orthogonal components

$$\int_{\tau}^{\tau+kT}\sin(m\omega t+\theta_m)\sin(\omega t+\theta_0)dt=0$$

Orthogonal components

$$\int_{\tau}^{\tau+kT}\sin(m\omega t+\theta_m)\sin(n\omega t+\theta_m)dt$$
$$=\int_{\tau}^{\tau+kT}\begin{pmatrix}\sin(m\omega t)\cos(\theta_m)\\+\cos(m\omega t)\sin(\theta_m)\end{pmatrix}\begin{pmatrix}\sin(n\omega t)\cos(\theta_n)\\+\cos(n\omega t)\sin(\theta_n)\end{pmatrix}dt$$
$$=\cos(\theta_m)\cos(\theta_n)\int_{\tau}^{\tau+kT}\sin(m\omega t)\sin(n\omega t)dt+$$
$$\cos(\theta_m)\sin(\theta_n)\int_{\tau}^{\tau+kT}\sin(m\omega t)\cos(n\omega t)dt+ \qquad (39)$$
$$\sin(\theta_m)\cos(\theta_n)\int_{\tau}^{\tau+T}\cos(m\omega t)\sin(n\omega t)dt+$$
$$\sin(\theta_m)\sin(\theta_n)\int_{\tau}^{\tau+kT}\cos(m\omega t)\cos(n\omega t)dt$$

The middle two integrals vanish because of the orthogonality, and the first and last integrals vanish if $m\neq n$, but if the voltage and current have the same harmonic order then:

$$\cos(\theta_m)\cos(\theta_n)\int_{\tau}^{\tau+kT}\sin^2(m\omega t)dt=\frac{kT\cos(\theta_m)\cos(\theta_n)}{2}$$

And

$$\sin(\theta_m)\sin(\theta_n)\int_{\tau}^{\tau+kT}\cos^2(m\omega t)dt=\frac{kT\sin(\theta_m)\sin(\theta_n)}{2}$$

Finally we have:

$$\int_{\tau}^{\tau+kT}\sin(m\omega t+\theta_m)\sin(n\omega t+\theta_n)dt=0 \text{ if } m\neq n$$
$$\int_{\tau}^{\tau+kT}\sin(m\omega t+\theta_m)\sin(k\omega t+\theta_n)dt$$
$$=\frac{kT\left(\cos(\theta_m)\cos(\theta_n)+\sin(\theta_m)\sin(\theta_n)\right)}{2} \qquad (40)$$
$$=\frac{kT\cos(\theta_n-\theta_m)}{2}$$
, if m, n have the same harmonic order

Substituting the above results into the equation for Watt:

$$P=V_0I_0\left(\cos(\theta_0)+a\beta\cos(\theta_n-\theta_m)\right)$$
And
$$\xi_P=\frac{a\beta\cos(\theta_n-\theta_m)}{\cos(\theta_0)}$$
if voltage and current have the same harmonic order.

4.3. Arithmetic VA in the presence distortion D_1

The arithmetic VA is determined from the RMS voltage and RMS current found in section 2.1:

$$S_A=V_0I_0\sqrt{\left(1+\beta^2\right)\left(1+a^2\right)}$$
And
$$\xi_{S_A}=\sqrt{\left(1+\beta^2\right)\left(1+a^2\right)}-1 \qquad (41)$$

4.4. Integral VAR in the presence distortion D_1

Given: the quadrature-phase component of voltage $v_\perp$ is computed by integration. The integral definition of VAR with the voltage and current functions given in section 4.0 above is evaluated as follows:

$$v_\perp(t)=\omega\sqrt{2}V_0\int\left(\sin(\omega t)+\beta\sin(m\omega t+\theta_m)\right)dt$$
$$=\omega\sqrt{2}V_0\left(\int\sin(\omega t)dt+\beta\int\sin(m\omega t+\theta_m)dt\right)$$
$$=\omega\sqrt{2}V_0\left(\frac{1}{\omega}\cos(\omega t)+\frac{\beta}{m\omega}\cos(m\omega t+\theta_m)\right)$$
$$=\sqrt{2}V_0\left(\cos(\omega t)+\frac{\beta}{m}\cos(m\omega t+\theta_m)\right)$$
$$Q_\int=\frac{1}{kT}\int_{\tau}^{\tau+kT}v_\perp i\,dt=$$
$$=\frac{2V_0I_0}{kT}\int_{\tau}^{\tau+kT}\left(\cos(\omega t)+\frac{\beta}{m}\cos(m\omega t+\theta_m)\right)\left(\sin(2\pi f_0t+\theta_0)+\alpha\sin(2\pi nf_0t+\theta_n)\right)dt$$
$$=\frac{2V_0I_0}{kT}\int_{\tau}^{\tau+kT}\cos(\omega t)\sin(\omega t+\theta_0)dt+$$
$$\frac{2V_0I_0\alpha}{kT}\int_{\tau}^{\tau+kT}\cos(\omega t)\sin(n\omega t+\theta_n)dt+$$

$$\frac{2V_0 I_0 \beta}{mkT}\int_{\tau}^{\tau+kT}\cos(m\omega t+\theta_m)\sin(\omega t+\theta_0)dt + \quad (42)$$
$$\frac{2V_0 I_0 \alpha\beta}{mkT}\int_{\tau}^{\tau+kT}\cos(m\omega t+\theta_m)\sin(n\omega t+\theta_n)dt$$

Evaluating each of these integrals:

$$\int_{\tau}^{\tau+kT}\cos(\omega t)\sin(\omega t+\theta_0)dt = \frac{kT\sin(\theta_0)}{2}$$

$\int_{\tau}^{\tau+kT}\cos(\omega t)\sin(n\omega t+\theta_n)dt = 0$ Orthogonal components

$\int_{\tau}^{\tau+kT}\cos(m\omega t+\theta_m)\sin(\omega t+\theta_0)dt = 0$ Orthogonal components

$$\int_{\tau}^{\tau+kT}\cos(m\omega t+\theta_m)\sin(n\omega t+\theta_n)dt$$
$$= \int_{\tau}^{\tau+kT}\left(\cos(m\omega t)\cos(\theta_m)-\sin(m\omega t)\sin(\theta_m)\right)\left(\sin(n\omega t)\cos(\theta_n)+\cos(n\omega t)\sin(\theta_n)\right)dt$$
$$= \cos(\theta_m)\cos(\theta_n)\int_{\tau}^{\tau+kT}\cos(m\omega t)\sin(m\omega t)dt +$$
$$\cos(\theta_m)\sin(\theta_n)\int_{\tau}^{\tau+kT}\cos(m\omega t)\cos(m\omega t)dt -$$
$$\sin(\theta_m)\cos(\theta_n)\int_{\tau}^{\tau+kT}\sin(m\omega t)\sin(m\omega t)dt -$$
$$\sin(\theta_m)\sin(\theta_n)\int_{\tau}^{\tau+kT}\sin(m\omega t)\cos(m\omega t)dt$$

The first and last integrals vanish because of the orthogonality, and the middle two integrals vanish if $m \neq n$, but if the voltage and current have the same harmonic order then:

$$\cos(\theta_m)\sin(\theta_n)\int_{\tau}^{\tau+kT}\cos^2(m\omega t)dt = \frac{kT\cos(\theta_m)\sin(\theta_n)}{2}$$

And

$$\sin(\theta_m)\cos(\theta_n)\int_{\tau}^{\tau+kT}\sin^2(m\omega t)dt = \frac{kT\sin(\theta_m)\cos(\theta_n)}{2}$$

Finally we have:

$\int_{\tau}^{\tau+kT}\cos(m\omega t+\theta_m)\sin(n\omega t+\theta_n)dt = 0$ if $m \neq n$

$$\int_{\tau}^{\tau+kT}\cos(m\omega t+\theta_m)\sin(n\omega t+\theta_n)dt$$
$$= \frac{kT\left(\sin(\theta_n)\cos(\theta_m)-\cos(\theta_n)\sin(\theta_m)\right)}{2}$$

$= \dfrac{kT\sin(\theta_n-\theta_m)}{2}$ if same harmonic order

Substituting the above results into the equation for VAR:

$Q_{\int} = V_0 I_0\left(\sin(\theta_0)+\dfrac{a\beta}{m}\sin(\theta_n-\theta_m)\right)$ And

$\xi_{Q_{\int}} = \dfrac{a\beta}{m}\dfrac{\sin(\theta_n-\theta_m)}{\sin(\theta_0)}$ if voltage and current have the same harmonic order.

4.5. Shifted VAR in the presence distortion D_1

Given: the quadrature-phase component of voltage $v_{\perp}$ is computed by 90 degrees shifting of the potential function. The shifted definition of VAR with the voltage and current functions given in section 4.0 above is evaluated as follows:

$$v_{\perp}(t) = v(t+\frac{\pi}{2\omega})$$
$$= \sqrt{2}V_0\left(\sin(\omega(t+\frac{\pi}{2\omega}))+\beta\sin(m\omega(t+\frac{\pi}{2\omega})+\theta_m)\right)$$
$$= \sqrt{2}V_0\left(\cos(\omega t)+\beta\sin(m\omega t+\theta_m+\frac{m\pi}{2})\right)$$
$$Q_{\pi/2} = \frac{1}{kT}\int_{\tau}^{\tau+kT}v_{\perp}i dt =$$
$$= \frac{2V_0 I_0}{kT}\int_{\tau}^{\tau+kT}\left(\cos(\omega t)+\beta\sin(m\omega t+\theta_m+\frac{m\pi}{2})\right)\left(\sin(2\pi f_0 t+\theta_0)+\alpha\sin(2\pi n f_0 t+\theta_n)\right)dt$$
$$= \frac{2V_0 I_0}{kT}\int_{\tau}^{\tau+kT}\cos(\omega t)\sin(\omega t+\theta_0)dt +$$
$$\frac{2V_0 I_0 \alpha}{kT}\int_{\tau}^{\tau+kT}\cos(\omega t)\sin(n\omega t+\theta_n)dt + \quad (43)$$
$$\frac{2V_0 I_0 \beta}{kT}\int_{\tau}^{\tau+kT}\sin(m\omega t+\theta_m+\frac{m\pi}{2})\sin(\omega t+\theta_0)dt +$$
$$\frac{2V_0 I_0 \alpha\beta}{kT}\int_{\tau}^{\tau+kT}\sin(m\omega t+\theta_m+\frac{m\pi}{2})\sin(n\omega t+\theta_n)dt$$

Evaluating each of these integrals:

$$\int_{\tau}^{\tau+kT}\cos(\omega t)\sin(\omega t+\theta_0)dt = \frac{kT\sin(\theta_0)}{2}$$

$\int_{\tau}^{\tau+kT}\cos(\omega t)\sin(n\omega t+\theta_n)dt = 0$ Orthogonal components

$$\int_{\tau}^{\tau+kT} \sin(m\omega t + \theta_m + \frac{m\pi}{2}) \sin(\omega t + \theta_0) dt = 0$$

Orthogonal components

$$\int_{\tau}^{\tau+kT} \sin(m\omega t + \theta_m + \frac{m\pi}{2}) \sin(n\omega t + \theta_n) dt$$

$$= \int_{\tau}^{\tau+kT} \begin{pmatrix} \sin(m\omega t)\cos(\theta_m + \frac{m\pi}{2}) \\ + \cos(m\omega t)\sin(\theta_m + \frac{m\pi}{2}) \end{pmatrix} \begin{pmatrix} \sin(n\omega t)\cos(\theta_n) \\ + \cos(n\omega t)\sin(\theta_n) \end{pmatrix} dt$$

$$= \cos(\theta_m + \frac{m\pi}{2})\cos(\theta_n)\int_{\tau}^{\tau+kT} \sin(m\omega t)\sin(m\omega t)dt +$$
$$\cos(\theta_m + \frac{m\pi}{2})\sin(\theta_n)\int_{\tau}^{\tau+kT} \sin(m\omega t)\cos(m\omega t)dt +$$
$$\sin(\theta_m + \frac{m\pi}{2})\cos(\theta_n)\int_{\tau}^{\tau+kT} \cos(m\omega t)\sin(m\omega t)dt +$$
$$\sin(\theta_m + \frac{m\pi}{2})\sin(\theta_n)\int_{\tau}^{\tau+kT} \cos(m\omega t)\cos(m\omega t)dt \quad (44)$$

The middle two integrals vanish because of the orthogonality, and the middle two integrals vanish if $m \neq n$, but if the voltage and current have the same harmonic order then:

$$\cos(\theta_m + \frac{m\pi}{2})\cos(\theta_n)\int_{\tau}^{\tau+kT} \sin^2(m\omega t)dt$$
$$= \frac{kT\cos(\theta_m + \frac{m\pi}{2})\cos(\theta_n)}{2}$$

and

$$\sin(\theta_m + \frac{m\pi}{2})\sin(\theta_n)\int_{\tau}^{\tau+kT} \cos^2(m\omega t)dt$$
$$= \frac{kT\sin(\theta_m + \frac{m\pi}{2})\sin(\theta_n)}{2}$$

Finally we have:

$\int_{\tau}^{\tau+kT} \sin(m\omega t + \theta_m + \frac{m\pi}{2}) \sin(n\omega t + \theta_n) dt = 0$ if $m \neq n$

$$\int_{\tau}^{\tau+kT} \sin(m\omega t + \theta_m + \frac{m\pi}{2}) \sin(n\omega t + \theta_n) dt$$
$$= \frac{kT\left(\cos(\theta_n)\cos(\theta_m + \frac{m\pi}{2}) + \sin(\theta_n)\sin(\theta_m + \frac{m\pi}{2})\right)}{2}$$
$$= \frac{kT\cos(\theta_n - \theta_m - \frac{m\pi}{2})}{2} \quad (45)$$

if same harmonic order

Substituting the above results into the equation for VAR:

$$Q_{\pi/2} = V_0 I_0 \left(\sin(\theta_0) + a\beta\cos(\theta_n - \theta_m - \frac{m\pi}{2}) \right) \quad (46)$$

and

$\xi_{Q_{\pi/2}} = \dfrac{a\beta\cos(\theta_n - \theta_m - \frac{m\pi}{2})}{\sin(\theta_0)}$ if voltage and current have the same harmonic order.

4.6. Vector VA in the presence of distortion D_1

Vector VA is a derived from the measurements of Watt and VAR. The defining equation for vector VA given in section 1.0 is:

$$S_{\rightarrow} = \sqrt{P^2 + Q^2} \quad (47)$$

If the harmonic order on the voltage axis is not equal to harmonic order on the current axis, there are no harmonic contributions in the active power in 4.2 or any contribution in the reactive power for 4.4 or 4.5. Therefore:

$S_{\rightarrow} = \sqrt{P_0^2 + Q_0^2}$ and $\xi_{S_{\rightarrow}} = 0$ (48)

4.6.1. Vector VA in the presence of distortion D_1: Integral VAR

If the harmonic order on the voltage axis is equal to harmonic order on the current axis then substituting the result derived in 4.2 and 4.4 above:

$$S_{\rightarrow} = V_0 I_0 \sqrt{\begin{array}{l} (\cos(\theta_0) + a\beta\cos(\theta_m - \theta_n))^2 \\ + \left(\sin(\theta_0) + \frac{a\beta}{m}\sin(\theta_n - \theta_m) \right)^2 \end{array}} \quad (49)$$

4.6.2. Vector VA in the presence of distortion D_1: Shifted VAR

If the harmonic order on the voltage axis is equal to harmonic order on the current axis then substituting the result derived in 4.2 and 4.5 above:

$$S_{\rightarrow} = V_0 I_0 \sqrt{\begin{array}{l}(\cos(\theta_0) + a\beta\cos(\theta_m - \theta_n))^2 + \\ \left(\sin(\theta_0) + a\beta\cos(\theta_n - \theta_m - \frac{m\pi}{2})\right)^2\end{array}} \quad (50)$$

4.7. RMS VAR in the presence of distortion D_1

Computing RMS VAR with respect to the power triangle yields:

A) If voltage and current have different harmonic orders:

$$\begin{aligned}Q_{rms} &= \sqrt{S_A{}^2 - P^2} \\ &= \sqrt{V_0^2 I_0^2 (1+\beta^2)(1+a^2) - V_0^2 I_0^2 \cos^2(\theta_0)} \\ Q_{rms} &= V_0 I_0 \sqrt{1+\beta^2 + a^2 + \alpha^2\beta^2 - \cos^2(\theta_0)} \\ &= V_0 I_0 \sqrt{\sin^2(\theta_0) + \beta^2 + a^2 + \alpha^2\beta^2}\end{aligned}$$

The error of the estimate would be:

$$\xi_{Q_{rms}} = \sqrt{1 + \csc^2(\theta_0)\left(\beta^2 + \alpha^2 + \beta^2\alpha^2\right)} - 1 \quad (51)$$

B) If voltage and current have different harmonic orders:

$$\begin{aligned}Q_{rms} &= \sqrt{S_A{}^2 - P^2} \\ &= \sqrt{\begin{array}{l}V_0^2 I_0^2 (1+\beta^2)(1+a^2) \\ -V_0^2 I_0^2 (\cos(\theta_0) + a\beta\cos(\theta_n - \theta_m))^2\end{array}} \\ &= V_0 I_0 \sqrt{\begin{array}{l}(1+\beta^2)(1+a^2) \\ -(\cos(\theta_0) + a\beta\cos(\theta_n - \theta_m))^2\end{array}} \\ &= V_0 I_0 \sqrt{\begin{array}{l}1+\alpha^2+\beta^2+\alpha^2\beta^2 \\ -\cos^2(\theta_0) - 2\alpha\beta\cos(\theta_0)\cos(\theta_n - \theta_m) \\ -a^2\beta^2\cos^2(\theta_n - \theta_m)\end{array}} \\ &= V_0 I_0 \sqrt{\begin{array}{l}\sin^2(\theta_0) + a^2\beta^2\sin^2(\theta_n - \theta_m) \\ +\alpha^2 + \beta^2 - 2\alpha\beta\cos(\theta_0)\cos(\theta_n - \theta_m)\end{array}}\end{aligned} \quad (52)$$

The error of the estimate would be:

$$\xi_{VAR} = \sqrt{1 + \csc^2(\theta_0)\begin{pmatrix}a^2\beta^2\sin^2(\theta_n - \theta_m) \\ +\alpha^2 + \beta^2 \\ -2\alpha\beta\cos(\theta_0)\cos(\theta_n - \theta_m)\end{pmatrix}} - 1 \quad (53)$$

4.8. Power Factor in the presence of distortion D_1

4.8.1. Power Factor in the presence of distortion D_1: Arithmetic VA

The Power factor can be derived from the arithmetic VA and Watt.

A) If the harmonic order on the voltage axis is not equal to harmonic order on the current axis, the power factor becomes:

$$\begin{aligned}PF_A &= \frac{P_0}{S_A} = \frac{V_0 I_0 \cos(\theta_0)}{V_0 I_0 \sqrt{(1+\beta^2)(1+a^2)}} \\ &= \frac{\cos(\theta_0)}{\sqrt{(1+\beta^2)(1+a^2)}}\end{aligned} \quad (54)$$

The error of the estimate would be:

$$\xi_{PF} = \frac{1}{\sqrt{(1+\beta^2)(1+a^2)}} - 1 \quad (55)$$

B) If the harmonic order on the voltage and current axis are equal, the power factor becomes:

$$\begin{aligned}PF_A &= \frac{P_0}{S_A} = \frac{V_0 I_0 (\cos(\theta_0) + a\beta\cos(\theta_n - \theta_m))}{V_0 I_0 \sqrt{(1+\beta^2)(1+a^2)}} \\ &= \frac{\cos(\theta_0) + a\beta\cos(\theta_n - \theta_m)}{\sqrt{(1+\beta^2)(1+a^2)}}\end{aligned} \quad (56)$$

The error of the estimate would be:

$$\xi_{PF_A} = \frac{1 + \alpha\beta\sec(\theta_0)\cos(\theta_n - \theta_m)}{\sqrt{(1+\beta^2)(1+a^2)}} - 1 \quad (57)$$

4.8.2. Power Factor in the presence of distortion D_1: Vector VA

The Power factor can be derived from the vector VA and Watt. If the harmonic order on the voltage axis is not equal to harmonic order on the current axis, the power factor becomes:

$PF_{\rightarrow} = \frac{P_0}{S_0} = \cos(\theta_0)$ and $\xi_{PF_{\rightarrow}} = 0$, no error in the estimate.

4.8.2.1. Power Factor in the presence of distortion D_1: Vector VA, Integral VAR

If the harmonic order on the voltage and current axis are equal, and Integral VAR was used the power factor becomes:

$$PF_{\rightarrow} = \frac{P}{S_{\rightarrow}} = \frac{\cos(\theta_0) + a\beta\cos(\theta_n - \theta_m)}{\sqrt{\left(\cos(\theta_0) + a\beta\cos(\theta_m - \theta_n)\right)^2 + \left(\sin(\theta_0) + \frac{a\beta}{m}\sin(\theta_n - \theta_m)\right)^2}} \quad (58)$$

The error of the estimate would be:

$$\xi_{PF_{\rightarrow}} = \frac{1 + a\beta\sec(\theta_0)\cos(\theta_n - \theta_m)}{\sqrt{\left(\cos(\theta_0) + a\beta\cos(\theta_m - \theta_n)\right)^2 + \left(\sin(\theta_0) + \frac{a\beta}{m}\sin(\theta_n - \theta_m)\right)^2}} - 1 \quad (59)$$

4.8.2.2. Power Factor in the presence of distortion D_1: Vector VA, Shifted VAR

If the harmonic order on the voltage and current axis are equal, and Shifted VAR was used the power factor becomes:

$$PF_{\rightarrow} = \frac{P}{S_{\rightarrow}} = \frac{\cos(\theta_0) + a\beta\cos(\theta_n - \theta_m)}{\sqrt{\left(\cos(\theta_0) + a\beta\cos(\theta_m - \theta_n)\right)^2 + \left(\sin(\theta_0) + a\beta\cos(\theta_n - \theta_m - \frac{m\pi}{2})\right)^2}} \quad (60)$$

The error of the estimate would be:

$$\xi_{PF_{\rightarrow}} = \frac{1 + a\beta\sec(\theta_0)\cos(\theta_n - \theta_m)}{\sqrt{\left(\cos(\theta_0) + a\beta\cos(\theta_m - \theta_n)\right)^2 + \left(\sin(\theta_0) + a\beta\cos(\theta_n - \theta_m - \frac{m\pi}{2})\right)^2}} - 1 \quad (61)$$

4.9. Conclusions regarding distortion D_1: A m[th] order on voltage and a n[th] order on current

Just as in the case of zero-mean additive Gaussian noise, a harmonic on the current or voltage axis will force the measurement device to include the harmonic magnitude in its RMS voltage and current measurements.

It is recognized that a orthogonal relationship exists between the fundamental and harmonic components:

$$\int_{\tau}^{\tau+kT} f_j f_k dt = \begin{cases} R \rightarrow j = k \\ 0 \rightarrow j \neq k \end{cases}$$

If the harmonic order on the voltage and current axis are not equal, the resulting metric estimates are analogous to that obtain for zero-mean additive Gaussian noise.

If the harmonic order on the voltage and current axis are equal, then a harmonic error component will appear on the active power estimate (watt), and the reactive power estimate (VAR).

5. Conclusion

VAR Algorithm Comparison:

- **VAR RMS** – Contains noise distortion error.
- **VAR Integral** – Contains error due to attenuated voltage and current harmonic contribution of $\frac{1}{kT}$.
- **VAR Shifted** – Contains error due even harmonic phase distortion whereas the even harmonic may not contribute to the overall calculation.

PF Algorithm Comparison:

- **PF with VAR RMS** – Contains error due to over estimated VAR content.
- **PF with Integral VAR**– Contains error from under estimated VAR content due to Attenuated voltage and current harmonic contribution of $\frac{1}{kT}$.
- **PF With Shifted VAR -** Contains error from under estimated VAR content due to even harmonic phase distortion.

VA Algorithm Comparison:

- **Arithmetic VA** – Contains noise distortion error.
- **Vector VA with Integral VAR** – Contains error from under estimated VAR content due to attenuated voltage and current harmonic contribution of $\frac{1}{kT}$
- **Vector VA with Shifted VAR** – Contains error from under estimated VAR content due to even harmonic phase distortion.

As shown above, each independent algorithm has benefits that vary depending on the quality of voltage and current being supplied and measured. It is also evident; when measurements via RMS methods are used to derive VA and VAR they have inherent error included that gets compiled into associated derived measurements.

From an algorithm error aspect the most accurate approach involves algorithms that derive its measurement using shifted integration in order to eliminate the non-orthogonal noise contribution.

Additionally, it shows that if Watt and VAR can be derived independently the error can be reduced on all associated derived measurements.

Unfortunately the measurement approaches are not consistent between devices and only when the methods are defined can the device accuracy be compared to a reference.

AKNOLEDGMENTS

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